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A Smallest Graph of Girth 5 and Valency 6

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1. INTRODUCTION

In [3], the problem of finding a regular graph G, of given girth n and valency $k \geqslant 3$, that has the least possible number $f(n, k)$ of vertices was discussed. If $1 \leqslant n \leqslant 4$, the solution is straightforward. If $n \geqslant 5$, only a few cases have been solved, though their existence has been proved by Erdös and Sachs (see [3, p. 82]). In [2], Robertson has shown that $f(5, 4) = 19$ and in [4], Wegner has shown that $f(5, 5) = 30$. In this paper, we give a graph of girth 5 and valency 6. We also show that $f(5, 6) = 40$.

2. A GRAPH OF GIRTH 5 AND VALENCY 6

A graph of girth 5 and valency 6 having 40 vertices is shown in Figure 2.1. (The referee informs us that N. Robertson exhibits in his thesis (Neil Robertson, Ph. D. Thesis, University of Waterloo, Waterloo, Canada) the same example in a different manner.)

Notation. If two vertices x and y in a graph are adjacent, we write $x \sim y$.

We shall show that a graph of girth 5 and valency 6 having fewer than 40 vertices does not exist. In fact, let G be a graph of girth 5 and valency 6. It is already known that G must have more than 37 vertices (see [1]). Suppose G has 38 vertices. Then any edge of G is contained in $5 \times 5 - 1 = 24$ pentagons so that, G must contain $\frac{1}{5}(24 \times 38 \times \frac{6}{2})$ pentagons which is impossible.

Next, suppose G has 39 vertices. We arrange them as in Figure 2.2.

Suppose for an arbitrary vertex C of G, the vertices A and B in Figure 2.2 are separated. Then any edge adjacent to C must be contained in $5 \times 5 - 2 = 23$ pentagons. But then G must contain $\frac{1}{5}(23 \times 39 \times \frac{6}{2})$ pentagons, which is impossible. Therefore for some vertex C, A and B must be adjacent.

We say a vertex belongs to set (n), if it is adjacent to the vertex (n) and has distance two from the vertex C ($n = 1, 2, ..., 6$). Any vertex (other than B) adjacent to A is called an A-vertex. The definition of a B-vertex is similar.

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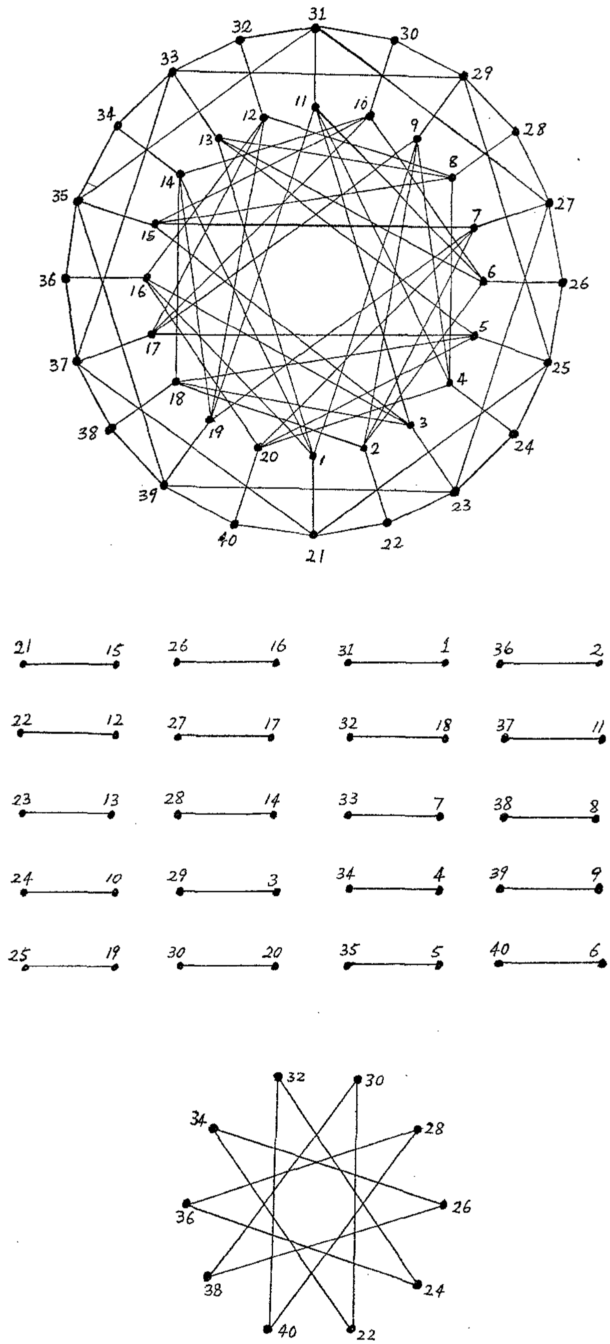
31 30 29 28 27 26 25 24 23 22 21 40 39 38 37 36 35 34 33 32
11 10 9 8 7 6 5 4 3 2 1 20 19 18 17 16 15 14 13 12
21 15
26 16
31 1
36 2
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37 11
23 13
28 14
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29 3
34 4
39 9
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30 20
35 5
40 6
32 30 28 26 24 22 40 38 36 34

FIGURE 2.1

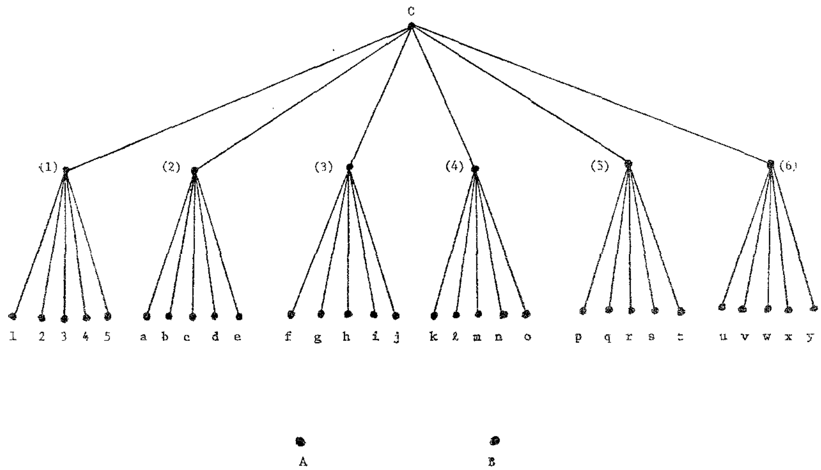

FIGURE 2.2

A-vertices and B-vertices are called end-vertices. A vertex having distance two from the vertex C is called an inner-vertex, if it is not an end-vertex.

We show that each inner-vertex must be adjacent to one A-vertex and one B-vertex. In fact, since $A \sim B$, no end-vertex can be adjacent to any other end vertex and no inner-vertex can be adjacent to more than one end-vertex of the same type. There are ten end-vertices and each must be adjacent to four inner-vertices. There are exactly twenty inner-vertices. Since each inner-vertex can be adjacent to at most two end-vertices, each must be adjacent to exactly one A-vertex and one B-vertex.

Case 1. Suppose A is adjacent to vertices a, f, k, p and u and B to vertices $5, e, j, o$ and t.

Case 1.1. Assume u is not joined to set (1). Then 5 is not joined to set (6). In fact, suppose 5 is joined to set (6). Since the valency of each vertex of G is six, the vertices 1, 2, 3 and 4 must be joined to set (6). But this is impossible, because u is not joined to set (1). Therefore 5 is not joined to set (6). Since u is an end-vertex, it must be adjacent to inner-vertices of sets (2), (3), (4) and (5). Assume arbitrarily that $u \sim b$, $x \sim d$, $2 \sim d$ and $3 \sim x$. Then d must be adjacent to two end-vertices from sets (3), (4) and (5) as must 2. Also 3 must be adjacent to at least one end-vertex from sets (3), (4) and (5). These five end-vertices are distinct. Clearly x cannot be adjacent to any of these five end-vertices. But x must be adjacent to two end-vertices from sets (3), (4) and (5) which is impossible.

Case 1.2. Assume $u \sim 1$. Then we can assume arbitrary that $v \sim 2$, $w \sim 3$, $x \sim 4$, $y \sim 5$ and $1 \sim t$.

Suppose $2 \sim p$, $2 \sim o$, $3 \sim k$, $3 \sim j$, $4 \sim f$ and $4 \sim e$. Then we can assume arbitrarily that $b \sim 1$, $c \sim 2$ and $d \sim 3$. Since 5 cannot be joined to set (2), 5 must be joined to sets (3), (4), (5) and (6). Therefore vertices 1, 2, 3, 4 and 5 are joined to the vertices of set (3) distinctively. Similarly they are joined to sets (4), (5) and (6), respectively. Also c must be joined to four vertices from sets (3), (4), (5) and (6). Since $c \sim 2$, these four vertices must be joined to vertices 1, 3, 4 and 5 respectively. Therefore it follows that vertex c must be joined to some vertex which is adjacent to 3. By a similar argument to that in Case 1.1, we see that it is impossible for c to be adjacent to w or to such an inner vertex in set (5). Now suppose $c \sim k$. Since $c \sim 2$ and $k \sim 3$, c must be adjacent to t. Since c must be joined to some vertex in set (3) which is adjacent to 5, $c \sim x$. Then it is impossible for x to be joined to two end-vertices without making a 4-circuit. If $c \sim j$, then a similar contradiction occures.

Suppose $2 \sim p, 2 \sim o, 3 \sim j, 4 \sim f, 4 \sim e$ and $3 \sim a$. Then we can assume arbitrarily that $1 \sim l$, $3 \sim m$ and $4 \sim n$. Vertex n must be joined to some vertex adjacent to 3, which is impossible by a similar argument as before.

Finally, if $2 \sim o$, $3 \sim k$, $3 \sim j$, $4 \sim f$, $4 \sim e$ and $2 \sim a$, a similar contradiction also occures. Therefore Case 1 is impossible.

Case 2. Assume A is adjacent to vertices a, f, k, p and u and B is adjacent to vertices e, j, o, t and y. Vertex b must be adjacent to two end-vertices and three inner-vertices. Assume arbitrarily that b is adjacent to 2, g, k, t and x, and also that $g \sim 3$, $k \sim 5$, $t \sim 4$ and $x \sim 1$.

Case 2.1. Suppose 3 is not adjacent to either end-vertex of set (2). Then of the six end-vertices of set (4), (5) and (6), five of them must be adjacent to b, 3 and 2. But g must be adjacent to two end-vertices (from these six end-vertices) which are not adjacent to b, 3 or 2 and this is impossible. We get a similar contradiction if 1 is not adjacent to an end-vertex of set (2) (we use vertex x).

Case 2.2. Assume 1 and 3 are adjacent to end-vertices of set (2), say $1 \sim a$ and $3 \sim e$. Suppose 4 and 5 are adjacent to end-vertices of set (6), then clearly $4 \sim u$ and $5 \sim y$. Also necessarily $3 \sim p$, $2 \sim o$, $2 \sim f$ and $1 \sim j$. Vertex x must be adjacent to two end-vertices from sets (3), (4) and (5). But x cannot be adjacent to any end-vertex which is adjacent to 1, 2 or b which is impossible. If 3 and 2 are adjacent to end-vertices of set (6), by a similar argument, g cannot be adjacent to two end-vertices from sets (4), (5) and (6) (because $2 \sim p$ and $2 \sim y$). Similarly, 3 and 5 cannot be adjacent to end-vertices of set (6).

Suppose 2 and 5 are adjacent to end-vertices of set (6), then clearly $5 \sim y$ and $2 \sim u$. Since 1 and 2 are adjacent to o and j, x must be adjacent to f and p.

But f and p are A-vertices and so this is impossible. Clearly 3 and 4 cannot be adjacent to end-vertices of set (6).

Finally assume 2 and 4 are adjacent to end-vertices of set (6), then clearly $4 \sim u$ and $2 \sim y$. Thus $3 \sim p$, $5 \sim j$, $2 \sim f$ and $1 \sim o$. Since x cannot be adjacent to any vertex which is joined to 1, 2 or b, $x \sim p$ and $x \sim j$. Similarly $g \sim o$ and $g \sim u$. Assume arbitrarily that $c \sim 4$ and $d \sim 5$. If $c \sim o$ and $c \sim p$, then $d \sim y$ and $d \sim f$. But $2 \sim y$ and $2 \sim f$ which is impossible. Also c cannot be adjacent to j. In fact, suppose $c \sim j$. Since $j \sim 5$ and $j \sim x$, c cannot be adjacent to k or p. Therefore $c \sim u$. But $c \sim 4$ and $u \sim 4$ which is impossible. Hence $c \nsim j$. Suppose $c \sim y$ and $c \sim p$. Since 1, 2, 3, 4 and 5 are joined to set (4), l, m and n must be adjacent to 2, 3 and 4, respectively. Since $y \sim 2$, $y \sim c$, $p \sim 3$, $p \sim c$ and $c \sim 4$, it follows that c cannot be adjacent to any inner vertex in set (4). Therefore $c \sim o$ and $c \sim f$. Then c must be joined to inner vertices from sets (5) and (6), which are adjacent to 3 and 5. Assume $5 \sim q$. Since 1, 2 and 5 are adjacent to the inner-vertices of set (5), $c \sim q$. But then q can only be adjacent to one end-vertex y which is a contradiction. Therefore G cannot have 39 vertices and this completes the proof.

REFERENCES

1. A. J. HOFFMAN AND R. R. SINGLETON, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* **4** (1960), 487–504.
2. N. ROBERTSON, The smallest graph of girth 5 and valency 4, *Bull. Amer. Math. Soc.* **70** (1964), 824–825.
3. W. T. TUTTE, "Connectivity in Graphs," Univ. of Toronto Press, Toronto, 1966.
4. G. WEGNER, A smallest graph of girth 5 and valency 5, *J. Combinatorial Theory* **14** (1973), 203–208.